Formules de trigonométrie hyperbolique

Soient $a, b, p, q, x, y \in \mathbb{R}$ (tels que les fonctions soient **bien définies**) et $n \in \mathbb{N}$.
La parfaite connaissance des graphes des fonctions trigonométriques est nécessaire.

Relations fondamentales

$$\operatorname{ch}^2(x) - \operatorname{sh}^2(x) = 1 \qquad \frac{d}{dx}\coth(x) = 1 - \coth^2(x) = -\frac{1}{\operatorname{sh}^2(x)} \qquad \frac{d}{dx}\operatorname{th}(x) = 1 - \operatorname{th}^2(x) = \frac{1}{\operatorname{ch}^2(x)}$$

$$\frac{d}{dx}\operatorname{Argch}(x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{Argsh}(x) = \frac{1}{\sqrt{x^2+1}} \qquad \frac{d}{dx}\operatorname{Argth}(x) = \frac{1}{1-x^2}$$

$$\operatorname{Argch}(x) = \ln\left(x + \sqrt{x^2-1}\right) \qquad \operatorname{Argsh}(x) = \ln\left(x + \sqrt{x^2+1}\right) \qquad \operatorname{Argth}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Il faut savoir linéariser et développer à l'aide des formules $(\operatorname{ch}(x) + \operatorname{sh}(x))^n = \operatorname{ch}(nx) + \operatorname{sh}(nx)$

$$\operatorname{ch}(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sh}(x) = \frac{e^x - e^{-x}}{2} \qquad e^x = \operatorname{ch}(x) + \operatorname{sh}(x) \qquad e^{-x} = \operatorname{ch}(x) - \operatorname{sh}(x).$$

Formules d'addition

$$\operatorname{ch}(a+b) = \operatorname{ch}(a)\operatorname{ch}(b) + \operatorname{sh}(a)\operatorname{sh}(b) \qquad \operatorname{ch}(a-b) = \operatorname{ch}(a)\operatorname{ch}(b) - \operatorname{sh}(a)\operatorname{sh}(b)$$

$$\operatorname{sh}(a+b) = \operatorname{sh}(a)\operatorname{ch}(b) + \operatorname{ch}(a)\operatorname{sh}(b) \qquad \operatorname{sh}(a-b) = \operatorname{sh}(a)\operatorname{ch}(b) - \operatorname{ch}(a)\operatorname{sh}(b)$$

$$\operatorname{th}(a+b) = \frac{\operatorname{th}(a)+\operatorname{th}(b)}{1+\operatorname{th}(a)\operatorname{th}(b)} \qquad \operatorname{th}(a-b) = \frac{\operatorname{th}(a)-\operatorname{th}(b)}{1-\operatorname{th}(a)\operatorname{th}(b)}$$

Formules d'angle double

$$\operatorname{ch}(2x) = \operatorname{ch}^2(x) + \operatorname{sh}^2(x) = 2\operatorname{ch}^2(x) - 1 = 2\operatorname{sh}^2(x) + 1$$

$$\operatorname{sh}(2x) = 2\operatorname{sh}(x)\operatorname{ch}(x) \qquad \operatorname{th}(2x) = \frac{2\operatorname{th}(x)}{1+\operatorname{th}^2(x)}$$

Formules du demi-angle

$$\operatorname{ch}^2(x) = \frac{1+\operatorname{ch}(2x)}{2} \qquad \operatorname{sh}^2(x) = \frac{\operatorname{ch}(2x)-1}{2} \qquad \operatorname{th}(x) = \frac{\operatorname{sh}(2x)}{1+\operatorname{ch}(2x)} = \frac{\operatorname{ch}(2x)-1}{\operatorname{sh}(2x)}$$

En posant $t = \operatorname{th}(\frac{x}{2})$, on a : $\operatorname{ch}(x) = \frac{1+t^2}{1-t^2}$, $\operatorname{sh}(x) = \frac{2t}{1-t^2}$ et $\operatorname{th}(x) = \frac{2t}{1+t^2}$.

Somme, différence et produit

$$\operatorname{ch}(p) + \operatorname{ch}(q) = 2\operatorname{ch}\left(\frac{p+q}{2}\right)\operatorname{ch}\left(\frac{p-q}{2}\right) \qquad \operatorname{ch}(p) - \operatorname{ch}(q) = 2\operatorname{sh}\left(\frac{p+q}{2}\right)\operatorname{sh}\left(\frac{p-q}{2}\right)$$

$$\operatorname{sh}(p) + \operatorname{sh}(q) = 2\operatorname{sh}\left(\frac{p+q}{2}\right)\operatorname{ch}\left(\frac{p-q}{2}\right) \qquad \operatorname{sh}(p) - \operatorname{sh}(q) = 2\operatorname{ch}\left(\frac{p+q}{2}\right)\operatorname{sh}\left(\frac{p-q}{2}\right)$$

$$\operatorname{th}(p) + \operatorname{th}(q) = \frac{\operatorname{sh}(p+q)}{\operatorname{ch}(p)\operatorname{ch}(q)} \qquad \operatorname{th}(p) - \operatorname{th}(q) = \frac{\operatorname{sh}(p-q)}{\operatorname{ch}(p)\operatorname{ch}(q)}$$

Procédé mnémotechnique : retenir « coco-sisi-sico-cosi » pour l'ordre des fonctions.

Les produits $\operatorname{ch}(a)\operatorname{ch}(b)$, $\operatorname{sh}(a)\operatorname{sh}(b)$ et $\operatorname{sh}(a)\operatorname{ch}(b)$ s'obtiennent à partir des formules d'addition.

Quelques autres formules

$$\int \operatorname{th}(x)\,dx = \ln(\operatorname{ch}(x)) \qquad \int \frac{1}{\operatorname{ch}(x)}\,dx = 2\operatorname{Arctan}(e^x)$$

$$\int \coth(x)\,dx = \ln(|\operatorname{sh}(x)|) \qquad \int \frac{1}{\operatorname{sh}(x)}\,dx = \ln\left(\left|\operatorname{th}\left(\frac{x}{2}\right)\right|\right)$$

Au voisinage de 0 :

$$\operatorname{ch}(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots + \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)$$

$$\operatorname{sh}(x) = x + \frac{x^3}{6} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right)$$

$$\operatorname{th}(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + o\left(x^8\right).$$